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ANALYSIS OF ULTRASONIC WAVE SCATTERING FOR THE CHARACTERIZATION OF DEFECTS IN SOLIDS

Final Report for the Period 1 Feb 74-15 Mar 77
February 1, 1974 through March 15, 1977

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

Measurements and analysis have been carried out on the scattering of ultrasonic elastic waves by voids and solid inclusions in titanium alloy. Both direct scattered amplitudes and mode converted scattering amplitudes were measured for incident shear and incident compressional plane waves. The scatterers were spherical and ellipsoidal in shape. Exact calculation for spherical shapes of arbitrary radius and elastic properties were obtained and compared with the experimental measurements. For other shapes, exact calculations are either

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20. <u>Abstract</u> (Cont'd)

impossible or impractical. Scattering from an arbitrary shape was expressed in terms of an integral equation from which an improved Born approximation reduces to the Rayleigh limit for scattering by spheres at low frequencies and to the known static stress field in ellipsoids for zero frequencies. The use of a variational approach has been explored which appears to have the capability of significantly extending the range of validity of the usual long wavelength limit approximations.

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I. INTRODUCTION

This report is the final report on Contract No. F44620-74-C-0057 to study the application of ultrasonic elastic wave scattering to the detection, identification and characterization of defects in solids. The contract was initiated on February 1, 1974, and terminated on March 15, 1977. During the three-year period it entailed approximately 1-1/2 man-years of effort of which approximately one-third was experimental measurement and two-thirds theoretical analysis. During the first two years the theoretical analysis was compared with experimental measurements, but it became apparent that this approach was not adequately providing a validation of the calculations since any discrepancies could be ascribed to various experimental shortcomings such as instrumental sensitivity and angular resolution, lack of monochromaticity in the transducer pulses, variability in the physical parameters of the titanium alloys, and attenuation and scattering of the ultrasonic signals by metallurgical inhomogeneities (second phases). For this reason the effort of the third year of the program was entirely analytic.

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II. SUMMARY OF PROGRESS

The results of the first two years have been reported in two annual reports [Cohen, 1975; Cohen, 1976] and an interim report [Cohen and Tittmann, 1975] and will only be summarized here. For details, the reader is referred to the prior reports which must be considered as part of the full documentation of this contract. The theoretical results on the formulation of the scattering of elastic waves will be reported here in some detail.

A. Scattering of Elastic Waves by Spherical Cavities and Inclusions

In order to verify the experimental techniques and to develop the necessary theoretical basis, the initial task of this contract was the review of the existing calculations of scattering by spheres [Cohen, 1975].

B. Scattering of Elastic Waves by Spherical Cavities

Spceimens of titanium alloy containing a spherical void of diameter 0.4 mm or 0.8 mm were prepared by diffusion bonding techniques. Experimental data on the scattering of ultrasonic pulses by these cavities were compared with the theoretical calculations. The data were in reasonable agreement with theory. Discrepancies could be ascribed to the lack of monochromaticity of the pulses at high frequencies, and to the scattering and attenuation of the pulses by the metallurgical inhomogeneities in the titanium alloy [Cohen, 1975].

C. Scattering of Compressional and Shear Waves by a Spherical Inclusion

A tungsten carbide inclusion (commercially-available tungsten carbide ball, nominal diameter, 1/32", 0.794 mm) was inserted into a diffusion-bonded titanium alloy block. Measurements of compressional wave scattering were carried out at nominal frequencies of 2.25, 5.0 and 10.0 MHz. The transducers

used had low Q (approximately 5) and hence not adequately monochromatic.
Because of this, waveform synthesis was required. The 2.25 MHz transducer
showed a characteristic with a full width at half maximum of 0.8 MHz. The
5 MHz transducers had a peak output at 4.5 MHz with a full width at half
maximum of 1.9 MHz, and the 10 MHz transducers peaked at 7.4 MHz with a full
width at half maximum of 2.6 MHz.

Experiments with incident shear waves were carried out with high Q shear-
cut quartz plates bonded to the faces of the sample.

These experiments were in agreement with theoretical calculations of the
cross sections except for the amplitude of cross-polarized (depolarized) shear
wave scattering at 5 MHz. The precise reasons for the anomalous appearance of
this depolarized amplitude (an obversed scattered shear wave polarized at
right angles to the scattering plane when the incident shear wave is parallel
to that plane, which is forbidden by symmetry for scattering by a sphere) is
not understood but has been ascribed to the bulk scattering by the metallurgical
second phase in the titanium alloy [Cohen, 1976].

D. Reformulation of the Calculation of Scattering by Spheres

The analyses of Ying and Truel [Ying and Truel, 1956] on the scattering of
compressional waves by spherical scatterers and of Einspruch, Witterholt and
Truel [Einspruch, et al., 1960] on the scattering of shear waves, were rederived
in a unified format which makes possible several useful extensions [Cohen and
Tittmann, 1976]:

a. Both scattering processes are expressed in terms of the same basic
   "scattering matrix" which allows the amplitudes of the spherical

harmonics of the scattered wave field to be expressed in terms of those of the incident wave field.

b. This same matrix applies also to the scattering of <u>any</u> incident wave field (not necessarily plane) and hence is in a convenient form to permit the analysis of near-field scattering or the multiple scattering of two spheres.

c. The reciprocity theorem for mode conversion scattering by a sphere:

$$\kappa^2 \sigma_{t \to \ell}(\theta) = k^2 {}_{\ell \to t}(\theta)$$

may be established directly from the symmetry of the "scattering matrix."

E. "Corrected" Born Approximation

The Born approximation is based on an integral equation formulation of the scattering process (see Section III.D below). The total displacement field is expressed as a sum of the imposed incident field plus an integral over the volume of the scatterer which involves the displacement field in the scatterer and the differences between the material constants of the scatterer and the constants of the external host medium. The first Born approximation replaces the displacement field in this integral by the incident displacement field. The resultant expression for the scattering cross section is an adequate approximation only when the properties of the scatterer differ slightly from those of the host medium. A correction to this first order Born approximation has been developed [Cohen, 1976] which yields the exact long-wavelength limit

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for the scattering cross sections for spherical scatterers for arbitrary variation of scatterer parameters. In particular, this corrected Born approximation gives the correct foward scattering amplitude for a spherical cavity in the Rayleigh limit.

A further extension of the Born approximation, which is developed in Section III.E below, gives results which are exact in the Rayleigh limit for scatterers of arbitrary shape (subject at most to the restriction of a smooth boundary).

F. Variational Formulation

From a variational formulation of the cross section, a further extension of the "corrected" Born approximation is possible. This includes the long-wavelength extension mentioned in the previous paragraph for $\omega \rightarrow 0$. The range of its applicability as a function of ω (or of ka, κa) may be partially evaluated by comparison with the known exact solution for a sphere. Application of this formulation to ellipsoids, in particular to ellipsoids of revolution, is underway. Further extensions which should increase the range of applicability of this class of approximation are possible without significantly increasing the complexity of the calculational algorithms. Details are given in Section III.F.

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III. DETAILS OF THIRD YEAR'S RESULTS--GREEN'S FUNCTION, RECIPROCITY AND THE BORN APPROXIMATION FOR THE SCATTERING OF ELASTIC WAVES

A. Differential Equation and Green's Function

The displacement field u_m in an elastic medium satisfies the equation

$$\rho \ddot{u}_m(\underline{r}) = \left(c_{mspj} u_{p,j}\right)_{,s} + f_m \tag{1}$$

where f_m is the applied force field. In Eq. (1) the elasticity tensor $c_{mspj}(\underline{r})$ may be a function of position. For an isotropic medium c_{mspj} is given by

$$c_{mpsj} = \lambda \delta_{ms} \delta_{pj} + \mu(\delta_{mp} \delta_{sj} + \delta_{mj} \delta_{sp}) \tag{2}$$

and for any arbitrary medium c_{mspj} must satisfy at least the symmetry with respect to interchange of indices expressed in Eq. (2); i.e., $c_{mspj} = c_{smpj} = c_{msjp} = c_{pjms}$.

For displacements which are harmonic in time with frequency ω Eq. (1) becomes

$$\left(c_{mspj} u_{p,j}\right)_{,s} + \rho \omega^2 u_m + f_m = 0 \quad . \tag{3}$$

The Green's function for the system is defined as the displacement field produced by a delta function impressed force at the point r_o, and hence $G_p^n(\underline{r};\underline{r}_o)$ is the solution of the equation

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$$\left(c_{mspj}G^n_{p,j}(\underline{r};\underline{r}_o)\right)_{,s} + \rho\omega^2 G^n_m(\underline{r};\underline{r}_o) + \delta_{mn}\delta(\underline{r}-\underline{r}_o) = 0 \tag{4}$$

The Green's function can in principle be defined for an inhomogeneous aniso-tropic system as the solution of Eq. (4), but it is useful for our purpose of exploring the scattering of elastic waves to restrict the Green's function to the solution of Eq. (4) for an infinite homogeneous isotropic medium, and hence to consider only the Green's function which is the solution of

$$(\lambda+\mu)G^n_{p,pm}(\underline{r};\underline{r}_o) + \mu G^n_{m,pp}(\underline{r};\underline{r}_o) + \rho\omega^2 G^n_m(\underline{r};\underline{r}_o) + \delta_{mn}\delta(\underline{r}-\underline{r}_o) = 0 \tag{5}$$

with the boundary condition

$$\lim_{|\underline{r}|\to\infty} G^n_m(\underline{r};\underline{r}_o) = 0 \quad . \tag{6}$$

It is clear that the solution of Eqs. (5) and (6) possess translational invariance and reciprocity:

$$G^n_m(\underline{r};\underline{r}_o) = G^n_m(|\underline{r}-\underline{r}_o|) = G^m_n(|\underline{r}_o-\underline{r}|) \quad , \tag{7}$$

The Green's function can be explicitly written [Morse and Feshbach, 1953; Gubernatis, et al., 1976]

$$G^j_m(|r-r_o|) = G_{jm}(R) = \frac{1}{4\pi\rho\omega^2}\left[\kappa^2\delta_{jm}\frac{e^{-i\kappa R}}{R} + \frac{\partial^2}{\partial x_j \partial x_m}\left\{\frac{e^{-i\kappa R}}{R} - \frac{e^{-ikR}}{R}\right\}\right] \tag{8}$$

We now introduce the spherical Bessel function $h_o(z)$ and write

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$$G_{jm}(R) = \frac{1}{4\pi i \rho \omega^2} \left[\kappa^3 \delta_{jm} h_o(\kappa R) + \frac{\partial^2}{\partial x_j \partial x_m} \left\{ \kappa h_o(\kappa R) - k h_o(kR) \right\} \right] \quad (9)$$

where κ and k are the wave numbers for shear and compressional waves,

$$\kappa^2 = \rho \omega^2 / \mu \quad , \quad k^2 = \rho \omega^2 / \lambda + 2\mu \quad . \quad (10)$$

and

$$h_o(z) = i e^{-iz}/z = j_o(z) - i y_o(z) \quad . \quad (11)$$

We shall also make use of the Bessel function relationships

$$h_{n-1}(z) + h_{n+1}(z) = \frac{(2n+1)}{z} h_n(z) \quad (12)$$

and, with $R^2 = x_1^2 + x_2^2 + x_3^2$,

$$\frac{\partial}{\partial x_j} \frac{h_n(kR)}{R^n} = -k x_j \frac{h_{n+1}(kR)}{R^{n+1}} \quad . \quad (13)$$

Using these expressions, we may write Eq. (9) in the form

$$G_{jm}(R) = \frac{1}{12\pi i \rho \omega^2} \left[\delta_{jm} \left\{ 2\kappa^3 h_o(\kappa R) + k^3 h_o(kR) \right\} \right.$$

$$\left. + \left(\frac{3 x_j x_m}{R^2} - \delta_{jm} \right) \left\{ \kappa^3 h_2(\kappa R) - k^3 h_2(kR) \right\} \right] \quad (14)$$

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The symmetry and reciprocity relationships of Eq. (7) are immediately evident in the explicit form of Eq. (8) or Eq. (14); however, a much more general reciprocity expression may be developed directly from Eq. (4) for the general case [Knopoff and Gangi, 1959; Cohen, 1976]. We rewrite Eq. (4) with a slightly different notation

$$\left(c_{mspj} u^{\nu}_{p,j}(\underline{r};\underline{r}_1)\right)_{,s} + \rho\omega^2 u^{\nu}_m(\underline{r};\underline{r}_1) + \delta_{m\nu}\delta(r-\underline{r}_1) = 0 \tag{15.1}$$

where $u^{\nu}_m(\underline{r};\underline{r}_1)$ is the displacement ar $\underline{r}$ resulting from a unit force in the ν-direction at $\underline{r}_1$, and

$$\left(c_{mspj} u^{\mu}_{p,j}(\underline{r};\underline{r}_2)\right)_{,s} + \rho\omega^2 u^{\mu}_m(\underline{r};\underline{r}_2) + \delta_{m\mu}\delta(\underline{r}-\underline{r}_2) = 0 \quad . \tag{15.2}$$

If (15.1) is multiplied by $u^{\mu}_m(\underline{r};\underline{r}_2)$ and (15.2) is multiplied by $u^{\nu}_m(\underline{r};\underline{r}_1)$ and the two expressions subtracted, one obtains

$$\int_v \left[\left(c_{mspj} u^{\nu}_{p,j}(\underline{r};\underline{r}_1)\right)_{,s} u^{\mu}_m(r;\underline{r}_2) - \left(c_{mspj} u^{\mu}_{p,j}(\underline{r};\underline{r}_2)\right)_{,s} u^{\nu}_m(\underline{r};\underline{r}_1)\right] d\underline{v}$$

$$= u^{\nu}_{\mu}(\underline{r}_2;\underline{r}_1) - u^{\mu}_{\nu}(\underline{r}_1;\underline{r}_2)$$

$$= \int_S c_{mspj}(\underline{r}) \left[u^{\mu}_m(\underline{r};\underline{r}_2) u^{\nu}_{p,j}(\underline{r};\underline{r}_1) - u^{\mu}_{p,j}(\underline{r};\underline{r}_2) u^{\nu}_m(\underline{r};\underline{r}_1)\right] \cdot d\underline{S}_s$$

$$+ \int_v c_{mspj}(\underline{r}) \left\{u^{\nu}_{m,s}(\underline{r};\underline{r}_1) u^{\mu}_{p,j}(\underline{r};\underline{r}_2) - u^{\nu}_{p,j}(\underline{r};\underline{r}_1) u^{\mu}_{m,s}(\underline{r};\underline{r}_2)\right\} d\underline{v}$$

$$\tag{16}$$

Now, from the symmetry of the elastic coefficients the final integral vanishes. If the region of integration extends to infinity the surface integral vanishes since the elastic displacements vanish (at least as fast as 1/R) as $|\underline{r}-\underline{r}_1|$ and $|\underline{r}-\underline{r}_2|$ go to infinity. If the region is finite we assume the usual homogeneous boundary conditions such that either the displacement, $u_m(r)$, or the normal stresses, $T_m = \tau_{ms} n_s = c_{mpsj} u_{p,j}(r) n_s$, vanish on the free surface.[*] In either case, one then obtains

$$u_\mu^\nu(\underline{r}_2;\underline{r}_1) = u_\nu^\mu(\underline{r}_1;\underline{r}_2) \tag{17}$$

Note that whereas the reciprocity of Eq. (8) applies only to an infinite isotropic homogeneous medium, the reciprocity of Eq. (17) is completely general.

B. The Integral Formulation of Scattering

The simple solution of Eq. (14) applies to an infinite homogeneous, isotropic medium. Although such a solution has only a limited idealized application per se, it can be used as the basis for an important extension of the analytic description of wave propagation in inhomogeneous elastic media. We assume that the medium of interest is isotropic and homogeneous for $R \rightarrow \infty$, but that in the vicinity of the origin there is an inhomogeneity which is the cause of scattering of elastic waves. We denote the "unperturbed" region by

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[*] The more general condition, that everywhere on the boundary we have $\alpha u_m + \beta \tau_{ms} n_s = 0$, in which α and β are arbitrary (but fixed) scalar parameters which may depend on position or on the component m, also leads to the vanishing of the surface integral point by point. The condition may even be further generalized to the extent that α or β may be taken to be symmetric matrices, although such generalization is probably physically artificial, except for the the case where α and β are diagonal.

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the density ρ^o and elastic constants c^o and write

$$\rho(\underline{r}) = \rho^o + \Delta\rho(\underline{r}) \tag{18.1}$$

$$c_{mspj}(\underline{r}) = c^o_{mspj} + \Delta c_{mspj} \tag{18.2}$$

The equation for the displacement field in the absence of applied forces is

$$\left(c_{mspj} u_{p,j}\right)_{,s} + \rho\omega^2 u_m(\underline{r}) = 0 \quad . \tag{19}$$

This is then written in the form

$$c^o_{mspj} u_{p,js}(\underline{r}) + \rho^o\omega^2 u_m(\underline{r}) + \left(\Delta c_{mspj} u_{p,j}\right)_{,s} + \Delta\rho\omega^2 u_m(\underline{r}) = 0 \quad . \tag{20}$$

When Eq. (20) is compared with Eq. (3), it is clear that the last two terms play the role of an impressed force on the medium (albeit one whose magnitude at each point is dependent upon the displacement field at that point). If c^o_{mspj} is given by Eq. (2) we can write a formal solution in terms of the Green's function for the homogeneous isotropic medium. We are interested in describing the waves scattered by the inhomogeneity; we therefore express $u_m(\underline{r})$ as the sum of the incident field $u^o_m(\underline{r})$, which is the wave field in the infinite homogeneous medium, and which satisfies the equation

$$c^o_{mspj} u^o_{p,js}(\underline{r}) + \rho^o\omega^2 u^o_m(\underline{r}) = 0 \quad , \tag{21}$$

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and a scattered wave, $u_m^{(s)}(\underline{r})$ generated by the inhomogeneity,

$$u_m(\underline{r}) = u_m^o(\underline{r}) + u_m^{(s)}(\underline{r}) \tag{22.1}$$

$$u_m^{(s)}(\underline{r}) = \int G_m^\nu(\underline{r};\underline{r}')\left[\left(\Delta c_{\nu spj}(\underline{r}')u_{p,j}(\underline{r}')\right)_{,s} + \Delta\rho(\underline{r}')\omega^2 u_\nu(\underline{r}')\right]d\underline{r}'$$

$$= \int G_{m\nu}(\underline{r}-\underline{r}')\left[\left(\Delta c_{\nu spj}(\underline{r}')u_{p,j}(\underline{r}')\right)_{,s}\right.$$

$$\left. + \omega^2\Delta\rho(\underline{r}')u_\nu(\underline{r}')\right]d\underline{r}' \tag{22.2}$$

The integral in Eq. (22.2) in principle extends over all space but since $\Delta c_{\nu spj}(\underline{r}')$ and $\Delta\rho(\underline{r})$ are such that they vanish as $|r'| \rightarrow \infty$ the integral in actuality is limited to a finite region of space. The scattering cross section, by definition, is evaluated in the far field (i.e., at a position which is far removed from the inhomogeneity which is scattering the elastic waves). In defining the scattering cross section one therefore needs the asymptotic behavior of the Bessel functions. These are

$$h_o(x) = \frac{ie^{-iz}}{z} \tag{23.1}$$

$$h_2(x) = \frac{ie^{-iz}}{z^3}\left[3 + 3iz - z^2\right] \tag{23.2}$$

and hence, from Eq. (14)

$$G_{jm}(R) = \frac{1}{4\pi\rho\omega^2 R}\left[\left(\delta_{jm} - \frac{x_j x_m}{R^2}\right)\kappa^2 e^{-i\kappa R} + \frac{x_j x_m}{R^2}k^2 e^{-ikR}\right] + O\left(\frac{1}{R^2}\right) \tag{24}$$

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and therefore with, $\underline{r} = r\underline{\Omega}$,

$$u_m^{(s)}(\underline{r}) = u_m^{(s)}(r\underline{\Omega}) = \frac{1}{r}\left[\kappa^2(\delta_{jm} - \Omega_j\Omega_m)e^{-i\kappa r}U_j(\kappa\underline{\Omega})\right.$$

$$\left. + k^2\Omega_j\Omega_m e^{-ikr}U_j(k\underline{\Omega})\right] + O\left(\frac{1}{r^2}\right) \qquad (25)$$

or

$$\underline{u}^{(s)}(\underline{r}) = \frac{1}{r}\left[k^2\underline{\Omega}\cdot\underline{U}(k\underline{\Omega})\underline{\Omega}e^{-ikr} + \kappa^2\underline{\Omega}\times\underline{U}(\kappa\underline{\Omega})\times\underline{\Omega}e^{-i\kappa r}\right] + O\left(\frac{1}{r^2}\right) \qquad (25.1)$$

and

$$U_j(q\underline{\Omega}) = \frac{1}{4\pi\rho\omega^2}\int e^{iq\underline{\Omega}\cdot\underline{r}'}\left\{\left(\Delta c_{jspn}(\underline{r}')u_{p,n}(\underline{r}')\right)_{,s}\right.$$

$$\left. + \omega^2\Delta\rho(\underline{r}')u_j(\underline{r}')\right\}d\underline{r}'$$

where q may be either κ or k. Then set $\underline{q} = q\underline{\Omega}$ and write

$$U_j(\underline{q}) = \frac{1}{4\pi\rho\omega^2}\int\left\{\omega^2\Delta\rho(\underline{r}')u_j(\underline{r}') - iq_s\Delta c_{sjpn}(\underline{r}')u_{p,n}(\underline{r}')\right\}e^{i\underline{q}\cdot\underline{r}'}d\underline{r}' \qquad (26)$$

The strain field $u_{j,s}$ can most easily be written in terms of the dilatation, $\varepsilon \equiv u_{s,s}$, and the "deviatrix," $E_{js} \equiv \frac{1}{2}(u_{j,s} + u_{s,j}) - \frac{1}{3}\varepsilon\delta_{js}$, which represents a pure shear; then

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$$c_{sjpn}u_{p,n} = c_{sjpn}E_{pn} + \frac{1}{3}c_{sjpp}\varepsilon \qquad (27.1)$$

which, for an isotropic medium, becomes

$$c_{sjpn}u_{p,n} = 2\mu E_{sj} + \left(\lambda + \frac{2}{3}\mu\right)\varepsilon\delta_{sj} \qquad (27.2)$$

and $\lambda + \frac{2}{3}\mu = K$ is the bulk modulus.

Equations (25) and (26) therefore give the scattering from a defect or inhomogeneity in an infinite homogeneous isotropic medium in terms of the displacement and strain fields within the inhomogeneity. To find $u_j(\underline{r}')$ and $u_{p,n}(\underline{r}')$, however, it is necessary to obtain the solution of the "internal" field defined by Eq. (22). Equation (22.2) is first integrated by parts to give

$$u_m(\underline{r}) = u_m^{(o)}(\underline{r}) + \omega^2 \int G_{mj}(\underline{r}-\underline{r}')\Delta\rho(\underline{r}')u_j(\underline{r}')d\underline{r}'$$

$$+ \int G_{mj,s}(\underline{r}-\underline{r}')\left\{\Delta c_{jspn}(\underline{r}')E_{pn}(\underline{r}') + \frac{1}{3}\Delta c_{jspp}(\underline{r}')\varepsilon(\underline{r}')\right\}d\underline{r}' \qquad (28)$$

C. Scattering Reciprocity

Equation (17) defines a reciprocity relationship between the displacement field at point $\underline{r}_1$ produced by an impressed force at $\underline{r}_2$ and the displacement field at $\underline{r}_2$ produced by an impressed force ar $\underline{r}_1$. It is also convenient to have a reciprocity condition which applies directly to the scattering cross section and the scattering vector defined in Eq. (25).

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We consider a scattering region near the origin of a coordinate system and two points $\underline{r}_1 = r_1\underline{\Omega}_1$ and $\underline{r}_2 = r_2\underline{\Omega}_2$. We assume that r_1 and r_2 are both large compared to the dimensions of the scattering region and with respect to the wavelengths of shear and compressional waves in the medium (Fig. 1). We describe the reciprocity statement of Eq. (17) to be composed of two components--the direct propagation of the wave field from point 1 to point 2 and a wave field scattered to point 2 by the scattering region. Then we can write

$$u_\mu^\nu(\underline{r}_2;\underline{r}_1) = G_\mu^\nu(|\underline{r}_2-\underline{r}_1|) + u_\mu^{(s)}(\underline{r}_2) \tag{29.1}$$

where $u_\mu^{(s)}$ is the displacement field at $\underline{r}_2$ which exists because of the presence of the scatterer. Without any loss of generality, one can describe $u_\mu^{(s)}(\underline{r}_2)$ as the result of a scattering of an incident wave by the scatterer,

$$u_\mu^{(s)}(\underline{r}_2) = G_\sigma^\nu(r_1)W_\mu^\sigma\left(\underline{r}_2;-\underline{\Omega}^{(1)},r_1\right) \tag{29.2}$$

where we have introduced the minus sign to indicate that the incident wave is moving in the $-\underline{\Omega}^{(1)}$ direction. Similarly

$$u_\nu^\mu(\underline{r}_1;\underline{r}_2) = G_\nu^\mu(|\underline{r}_1-\underline{r}_2|) + G_\sigma^\mu(r_2)W_\nu^\sigma\left(\underline{r}_1;-\underline{\Omega}^{(2)},r_2\right) \tag{30}$$

In writing Eqs. (29) and (30), we have done no more than define new quantities $W_\mu^\sigma\left(\underline{r}_2;-\underline{\Omega}^{(1)},r_1\right)$ and $W_\nu^\sigma\left(\underline{r}_1;-\underline{\Omega}^{(2)},r_2\right)$. We are interested, however, in the limiting case $r_1 \to \infty$, $r_2 \to \infty$, and in this limit $W_\mu^\sigma\left(\underline{r}_2;-\underline{\Omega}^{(1)},r_1\right)$ will be independent of the magnitude of $\underline{r}_1$ and similarly $W_\nu^\sigma\left(\underline{r}_1;-\underline{\Omega}^{(2)},r_2\right)$ will be independent

of r_2. Since Eq. (17) is valid for an arbitrary inhomogeneous elastic medium, it is also valid for the specialization to a localized inhomogeneity. Furthermore, the Green's function for the infinite homogeneous, isotropic medium is symmetric so that we have

$$G_\sigma^\nu(\underline{r}_1)W_\mu^\sigma\left(\underline{r}_2;-\underline{\Omega}^{(1)},r_2\right) = G_\sigma^\mu(r_2)W_\nu^\sigma\left(\underline{r}_1;-\underline{\Omega}^{(2)},r_2\right) \tag{31}$$

In the limit as r_1 and r_2 become large, Eq. (31) leads to

$$r_2\left[\kappa^2\left(\delta_{\nu\sigma} - \Omega_\nu^{(1)}\Omega_\sigma^{(1)}\right)e^{-i\kappa r_1} + k^2\Omega_\nu^{(1)}\Omega_\sigma^{(1)}e^{-ikr_1}\right]W_\mu^\sigma\left(\underline{r}_2;-\underline{\Omega}^{(1)},r_1\right)$$

$$= r_1\left[\kappa^2\left(\delta_\mu\delta_s - \Omega_\mu^{(2)}\Omega_s^{(2)}\right)e^{-i\kappa r_2} + k^2\Omega_\mu^{(2)}\Omega_s^{(2)}e^{-ikr_2}\right]W_\nu^s\left(\underline{r}_1;-\underline{\Omega}^{(2)},r_2\right) \tag{32}$$

Now $W_\mu^\sigma\left(\underline{r}_2;-\underline{\Omega}^{(1)}\right)$ and $W_\nu^s\left(\underline{r}_1;-\underline{\Omega}^{(2)},r_2\right)$ can be decomposed into longitudinal and transverse components. Ar large distances $W_\mu^\sigma\left(\underline{r}_2;-\underline{\Omega}^{(1)},r_1\right)$ must behave as

$$\lim_{r_1,r_2\to\infty} W_\mu^\sigma\left(\underline{r}_2;-\underline{\Omega}^{(1)},r_1\right)$$

$$= \frac{1}{r_2}\left\{\kappa^2\left(\delta_{\mu s} - \Omega_\mu^{(2)}\Omega_s^{(2)}\right)U_s^\sigma\left(\underline{\Omega}^{(2)},t;-\underline{\Omega}^{(1)}\right)e^{-i\kappa r_2}\right.$$

$$\left. + k^2\Omega_\mu^{(2)}\Omega_s^{(2)}U_s^\sigma\left(\underline{\Omega}^{(2)},\ell;-\underline{\Omega}^{(1)}\right)e^{-ikr_2}\right\} \tag{33}$$

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(where we have introduced the additional descriptor ℓ or t to emphasize the longitudinal or transverse components of the vector $\underline{U}^\sigma$), with a similar expression, of course, for $W^\sigma_\nu \left(\underline{r}_1; -\underline{\Omega}^{(2)}, r_2\right)$. Equation (32) can then be decomposed into the form

$$
\kappa^2\left(\delta_{\nu\sigma} - \Omega^{(1)}_\nu \Omega^{(1)}_\sigma\right)\left\{\kappa^2\left(\delta_{\mu s} - \Omega^{(2)}_\mu \Omega^{(2)}_s\right) U^\sigma_s\left(\underline{\Omega}^{(2)}, t; -\underline{\Omega}^{(1)}, t\right) e^{-i\kappa(r_1 + r_2)}\right.
$$

$$
\left. + k^2 \Omega^{(2)}_\mu \Omega^{(2)}_s U^\sigma_s\left(\underline{\Omega}^{(2)}, \ell; -\underline{\Omega}^{(1)}, t\right) e^{-i\kappa r_1 - ik r_2}\right\}
$$

$$
+ k^2 \Omega^{(1)}_\nu \Omega^{(1)}_\sigma \left\{\kappa^2\left(\delta_{\mu s} - \Omega^{(2)}_\mu \Omega^{(2)}_s\right) U^\sigma_s\left(\underline{\Omega}^{(2)}, t; -\underline{\Omega}^{(1)}, \ell\right) e^{-ik r_1 - i\kappa r_2}\right.
$$

$$
\left. + k^2 \Omega^{(2)}_\mu \Omega^{(2)}_s U^\sigma_s\left(\underline{\Omega}^{(2)}, \ell; -\underline{\Omega}^{(1)}, \ell\right) e^{-ik(r_1 + r_2)}\right\}
$$

$$
= \kappa^2\left(\delta_\mu \delta_s - \Omega^{(2)}_\mu \Omega^{(2)}_s\right)\left\{\kappa^2\left(\delta_{\nu\sigma} - \Omega^{(1)}_\nu \Omega^{(1)}_\sigma\right) U^s_\sigma\left(\underline{\Omega}^{(1)}, t; -\underline{\Omega}^{(2)}, t\right) e^{-i\kappa(r_1 + r_2)}\right.
$$

$$
\left. + k^2 \Omega^{(1)}_\nu \Omega^{(1)}_\sigma \left(U^s_\sigma \left(\underline{\Omega}^{(1)}, \ell; -\underline{\Omega}^{(2)}, t\right)\right) e^{-i\kappa r_2 - ik r_1}\right\}
$$

$$
+ k^2 \Omega^{(2)}_\mu \Omega^{(2)}_s \left\{\kappa^2\left(\delta_{\nu\sigma} - \Omega^{(1)}_\nu \Omega^{(1)}_\sigma\right) U^s_\sigma\left(\underline{\Omega}^{(1)}, t; -\underline{\Omega}^{(2)}, \ell\right) e^{-i\kappa r_1 - ik r_2}\right.
$$

$$
\left. + k^2 \Omega^{(1)}_\nu \Omega^{(1)}_\sigma U^s_\sigma\left(\underline{\Omega}^{(1)}, \ell; -\underline{\Omega}^{(2)}, \ell\right) e^{-ik(r_1 + r_2)}\right\} . \tag{34}
$$

From this we obtain the fundamental scattering reciprocity relations

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$$U_s^\sigma\left(\underline{\Omega}^{(2)},t;-\underline{\Omega}^{(1)},t\right) = U_\sigma^s\left(\underline{\Omega}^{(1)},t;-\underline{\Omega}^{(2)},t\right) \tag{35.1}$$

$$U_s^\sigma\left(\underline{\Omega}^{(2)},\ell;-\underline{\Omega}^{(1)},\ell\right) = U_\sigma^s\left(\underline{\Omega}^{(1)},\ell;-\underline{\Omega}^{(2)},\ell\right) \tag{35.2}$$

$$U_s^\sigma\left(\underline{\Omega}^{(2)},t;-\underline{\Omega}^{(1)},\ell\right) = U_\sigma^s\left(\underline{\Omega}^{(1)},\ell;-\underline{\Omega}^{(2)},t\right) \tag{35.3}$$

or simply,

$$U_s^\sigma\left(\underline{\Omega}^{(2)};-\underline{\Omega}^{(1)}\right) = U_\sigma^s\left(\underline{\Omega}^{(1)};-\underline{\Omega}^{(2)}\right) \tag{35}$$

(The fourth expression which can be obtained from Eq. (34) is equivalent to Eq. (35.3) with s and σ and (1) and (2) interchanged.)

The scattering amplitudes W_μ^σ defined in Eqs. (29) and (30) and their decomposition into longitudinal and transverse components U_s^σ defined in Eq. (33) refer specifically to the scattering associated with an incident field of unit amplitude. The incident wave will then carry an energy flux per unit area given by

$$Q^{in} = \rho\omega^3/q^{in} \tag{36.1}$$

where q^{in} is the propagation number (either k or κ) of the incident wave. The scattered energy flux per unit area normal to the direction $\underline{n}$ is given by

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$$Q_n = -Re\sigma_{nj}^* \dot{u}_j = \omega Im\sigma_{nj}^* u_j$$

$$= \omega Im\left[\left(\lambda + \frac{2}{3}\mu\right)\varepsilon^* u_n + 2\mu E_{nj}^* u_j\right] \tag{36.2}$$

In the far field we use Eq. (25) and obtain

$$\varepsilon = \frac{-ik^3}{r}\Omega_p U_p(k\underline{\Omega})e^{-ikr} + \bigcirc\left(\frac{1}{r^2}\right) \tag{37.1}$$

and

$$2E_{nj} = -\frac{i}{r}\left[\kappa^3(\Omega_n\delta_{js} + \Omega_j\delta_{ns} - 2\Omega_n\Omega_j\Omega_s)U_s(\kappa\underline{\Omega})e^{-i\kappa r}\right.$$

$$\left. + 2k^3\left(\Omega_n\Omega_j - \frac{1}{3}\delta_{nj}\right)\Omega_s U_s(k\underline{\Omega})e^{-ikr}\right] + \bigcirc\left(\frac{1}{r^2}\right) \tag{37.2}$$

The scattering cross section is defined as the far field scattered energy flux per unit solid angle divided by the incident plane wave energy flux per unit area

$$S(\underline{\Omega}) = \frac{1}{Q^{in}}\lim_{r\to\infty}r^2\Omega_n Q_n$$

$$= \frac{q^{in}}{\rho\omega^2}\lim_{r\to\infty}r^2 Im\left[\left(\lambda + \frac{2}{3}\mu\right)\varepsilon^*\Omega_m u_m + 2\mu\Omega_m E_{mj}^* u_j\right]$$

$$= q^{in}\left[k^3|\Omega_m U_m(k\underline{\Omega})|^2 + \kappa^3(\delta_{ms} - \Omega_m\Omega_s)ReU_m^*(\kappa\underline{\Omega})U_s(\kappa\underline{\Omega})\right]$$

$$= q^{in}\left[k^3|\Omega_m U_m(k\underline{\Omega})|^2 + \kappa^3|\underline{U}(\kappa\underline{\Omega})|^2 - \kappa^3|\Omega_m U_m(\kappa\underline{\Omega})|^2\right] \tag{38}$$

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which may be written

$$S(\underline{\Omega}) = q^{in}\left[k^3|\underline{\Omega}\cdot\underline{U}(k\underline{\Omega})|^2 + \kappa^3|\underline{\Omega}\times\underline{U}(\kappa\underline{\Omega})|^2\right] \tag{38.1}$$

It is clear from Eq. (38) that the vector $\underline{U}(\underline{q})$ contains all of the information in the displacement field $\underline{u}$ at large distances from the scattering center and contains it in a form which separates the purely geometrical aspects of the scattering from the properties of the scatterer.

It is useful to further decompose the scattering cross section into the shear and compressional wave scattering. We then write

$$S_{\ell\to\ell}(\underline{\Omega}^{sc};\underline{\Omega}^{in}) = k^4|\Omega_m^{sc}\Omega_\sigma^{in}U_{m\sigma}(k\underline{\Omega}^{sc};-k\underline{\Omega}^{in})|^2 \tag{39.1}$$

$$S_{\ell\to t}(\underline{\Omega}^{sc};\underline{\Omega}^{in}) = k\kappa^3|(\delta_{ms} - \Omega_m^{sc}\Omega_s^{sc})\Omega_\sigma^{in}U_{s\sigma}(\kappa\underline{\Omega}^{sc};-k\underline{\Omega}^{in})|^2 \tag{39.2}$$

$$S_{t\to\ell}(\underline{\Omega}^{sc};\underline{\Omega}^{in}) = \kappa k^3|\Omega_m^{sc}(\delta_{\nu\sigma} - \Omega_\nu^{in}\Omega_\sigma^{in})U_{m\nu}(k\underline{\Omega}^{sc};-\kappa\underline{\Omega}^{in})|^2 \tag{39.3}$$

$$S_{t\to t}(\underline{\Omega}^{sc};\underline{\Omega}^{in}) = \kappa^4|(\delta_{ms} - \Omega_m^{sc}\Omega_s^{sc})(\delta_{\nu\sigma} - \Omega_\nu^{in}\Omega_\sigma^{in})U_{s\nu}(k\underline{\Omega}^{sc};-\kappa\underline{\Omega}^{in})|^2 \tag{39.4}$$

and we obtain the reciprocity theorems for scattering

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$$S_{\ell \to \ell}(\underline{\Omega}_1; -\underline{\Omega}_2) = S_{\ell \to \ell}(\underline{\Omega}_2; -\underline{\Omega}_1) \tag{40.1}$$

$$S_{t \to t}(\underline{\Omega}_1; -\underline{\Omega}_2) = S_{t \to t}(\underline{\Omega}_2; -\underline{\Omega}_1) \tag{40.2}$$

$$k^2 S_{\ell \to t}(\underline{\Omega}_1; -\underline{\Omega}_2) = \kappa^2 S_{t \to \ell}(\underline{\Omega}_2; -\underline{\Omega}_1) \tag{40.3}$$

D. The Born Approximation

The scattering cross section may be calculated from Eqs. (22), (26) and (38.1). The Born approximation in its simplest form assumes that the scattered wave, $u_m^{(s)}(\underline{r})$ may be neglected in (26) and that $u_m(\underline{r})$ may be approximated by the incident wave $u_m^o(\underline{r})$. We now write the incident plane wave as

$$u_m^o(\underline{r}) = b_m e^{-i\underline{q}^{in} \cdot \underline{r}} \tag{41.1}$$

with the understanding that one characterizes the longitudinal and transverse waves by the relationships

longitudinal: $\quad q_m^{in} q_m^{in} = k^2 \quad , \quad b_m q_m^{in} = k \quad , \quad b_m b_m = 1 \tag{41.2}$

transverse: $\quad q_m^{in} q_m^{in} = \kappa^2 \quad , \quad b_m q_m^{in} = 0 \quad , \quad b_m b_m = 1 \tag{41.3}$

The scattering amplitudes in the Born approximation are therefore given by the following expressions [Cohen, 1976; Gubernatis, et al., 1976], when the scattering inhomogeneity is restricted to be isotropic:

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(a) Incident Longitudinal Wave

$$U_{\ell \to \ell} = \frac{f_\rho}{\rho} \cos\theta - \frac{1}{\lambda+2\mu}\left[f_d + 2f_\mu\left(\cos^2\theta - \frac{1}{3}\right)\right] \tag{42.1}$$

$$U_{\ell \to t} = \frac{f_\rho}{\rho} \sin\theta - \frac{k}{\kappa} \frac{f_\mu}{\mu} \sin2\theta \tag{42.2}$$

where θ is the angle of scattering, $\cos\theta = \Omega_j^{in}\Omega_j^{sc}$,

$$f_\rho(\underline{K}) = \frac{1}{4\pi}\int \Delta\rho(\underline{r})e^{i\underline{K}\cdot\underline{r}}d\underline{r} \quad ; \quad \underline{K} = \underline{q}^{sc} - \underline{q}^{in} \tag{43.1}$$

$$f_d(\underline{K}) = \frac{1}{4\pi}\int \left[\Delta\lambda(\underline{r}) + \frac{2}{3}\Delta\mu(\underline{r})\right]e^{i\underline{K}\cdot\underline{r}}d\underline{r} \tag{43.2}$$

$$f_\mu(\underline{K}) = \frac{1}{4\pi}\int \Delta\mu(\underline{r})e^{i\underline{K}\cdot\underline{r}}d\underline{r} \tag{43.3}$$

Hence, these functions are the Fourier transforms of the perturbation
of the density and elastic constants in the scatterer.

(b) Incident Transverse Wave (Plane Polarized)

$$U_{t \to \ell} = \cos\phi\left[\frac{f_\rho}{\rho}\sin\theta - \frac{k}{\kappa}\frac{f_\mu}{\mu}\sin2\theta\right] \tag{44.1}$$

$$U_{t \to t_\parallel} = \cos\phi\left[\frac{f_\rho}{\rho}\cos\theta - \frac{f_\mu}{\mu}\cos2\theta\right] \tag{44.2}$$

$$U_{t \to t_\perp} = -\sin\phi\left[\frac{f_\rho}{\rho} - \frac{f_\mu}{\mu}\cos\theta\right] \tag{44.3}$$

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where ϕ is the angle between the plane of polarization and the scattering plane, and $U_{t \to t_{\parallel}}$, $U_{t \to t_{\perp}}$ are the scattering components parallel and perpendicular, respectively, to the scattering plane.

(c) Incident Transverse Wave (Circular Polarization)

$$U_{t_r \to \ell} = \frac{1}{2} e^{-ir\phi}\left[\frac{f_\rho}{\rho} \sin\theta - \frac{k}{\kappa} \frac{f_\mu}{\mu} \sin2\theta\right] \qquad (45.1)$$

$$U_{t_r \to t_s} = \frac{1}{2} e^{-ir\phi}\left[(\cos\theta + rs) \frac{f_\rho}{\rho} - (\cos2\theta + rs\cos\theta) \frac{f_\mu}{\mu}\right] \qquad (45.2)$$

$$= \frac{1}{2} e^{-ir\phi}(\cos\theta + rs)\left[\frac{f_\rho}{\rho} - (2\cos\theta - rs) \frac{f_\mu}{\mu}\right]$$

where $r = \pm1$ gives the polarization of the incident wave and $s = \pm1$ that of the scattered wave. We note that for forward scattering there is no depolarization ($s = r$, only) while for backward scattering there is total depolarization ($s = -r$, only).

To extract the scattering cross section from the scattering amplitude one has

$$S(\Omega) = q^{in}q^{sc}|q^{sc}U|^2 \qquad (46)$$

E. Improved Born Approximation

The expressions given in the previous section are valid only for a limited range of parameters. In general, one must restrict their use to cases not only for which κa is small (where a is a characteristic dimension

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of the inhomogeneity) but also for which the magnitude of the perturbation, $|\Delta\rho(\underline{r})|/|\rho$, $|\Delta c(\underline{r})/c|$ is also small. This latter constraint is the more serious one since it means that the Born approximation is inapplicable to most problems of interest to NDE, where the most common scattering center is a void, for which $\Delta\rho = -\rho$. The Born approximation, in replacing $u_m(\underline{r})$ by $u_m^0(\underline{r})$ assumes that the integrals in Eq. (28) may be neglected. This is an acceptable approximation for $u_m(\underline{r})$ but it is not acceptable for its derivative or for $\varepsilon_{mj}(\underline{r})$. To see this we note that $u_{m,s}(\underline{r})$ will involve $G_{mj,ps}(\underline{r})$ which, as may be seen directly from Eq. (5), must contain delta-functions.

Specifically, we find, as extensions of Eq. (14)

$$G_{jm}(R) = \frac{1}{4\pi i \rho \omega^2} \left[\kappa^3 h_o(\kappa R)\delta_{jm} - \delta_{jm}\phi_1^{(o)}(R) + x_j x_m \phi_2^{(o)}(R) \right] \tag{47}$$

where

$$\phi_n^{(p)}(R) = \frac{1}{R^n} \left[\kappa^{n+p+1} h_n(\kappa R) - k^{n+p+1} h_n(kR) \right] \tag{48}$$

From Eqs. (12) and (13) one finds

$$\phi_n^{(p)}(R) = \frac{1}{2n+1} \left[R^2 \phi_{n+1}^{(p)}(R) + \phi_{n-1}^{(p+2)}(R) \right] \tag{49.1}$$

$$= \frac{1}{R^2} \left[(2n-1)\phi_n^{(p)}(R) - \phi_{n-2}^{(p+2)}(R) \right] \tag{49.2}$$

$$\frac{\partial \phi_n^{(p)}(R)}{\partial x_s} \equiv \phi_{n,s}^{(p)}(R) = -x_s \phi_{n+1}^{(p)} \tag{50}$$

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and

$$G_{jm,s}(R) = \frac{1}{4\pi i\rho\omega^2}\left[-\frac{\kappa^4}{R}h_1(\kappa R)x_s\delta_{jm} + (x_j\delta_{ms} + x_m\delta_{js} + x_s\delta_{jm})\phi_2^{(o)}(R)\right.$$

$$\left. -x_j x_m x_s \phi_3^{(o)}(R)\right] \tag{51.1}$$

$$G_{jm,sp} = \frac{1}{4\pi i\rho\omega^2}\left[\left(\frac{\kappa^4}{R}h_1(\kappa R)\delta_{sp} + \frac{\kappa^5}{R^2}h_2(\kappa R)x_s x_p\right)\delta_{jm}\right.$$

$$\left. + \Delta_{jmps}\phi_2^{(o)}(R) - X_{jmsp}\phi_s^{(o)}(R) + x_j x_m x_s x_p \phi_4^{(o)}(R)\right]$$

$$+ \frac{\delta(R)}{\rho\omega^2}\left[\frac{1}{15}(\kappa^2 - k^2)\Delta_{jmsp} - \frac{1}{3}\kappa^2\delta_{jm}\delta_{sp}\right]$$

$$= \frac{1}{12\pi i\rho\omega^2}\left[-\kappa^5 h_o(\kappa R)\delta_{jm}\delta_{sp} + \left(3\frac{x_s x_p}{R^2} - \delta_{sp}\right)\delta_{jm}\kappa^5 h_2(\kappa R)\right.$$

$$+ 3\left\{\Delta_{jmsp} - \frac{5X_{jmsp}}{R^2} + 35\frac{x_j x_m x_s x_p}{R^2}\right\}\phi_2^{(o)}(R)$$

$$+ \left\{X_{jmsp} - 10\frac{x_j x_m x_s x_p}{R^2}\right\}\phi_2^{(2)}(R)$$

$$+ \left\{\frac{X_{jmsp}}{R^2} - 7\frac{x_j x_m x_s x_p}{R^4}\right\}\phi_o^{(4)}(R)\right]$$

$$+ \frac{\delta(R)}{\rho\omega^2}\left[\frac{1}{15}(\kappa^2 - k^2)\Delta_{jmsp} - \frac{1}{3}\kappa^2\delta_{jm}\delta_{sp}\right] \tag{51.2}$$

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where the delta function is defined such that

$$R^3 \delta(R) = 0$$

$$4\pi \int \delta(R) R^2 dR = 1 \quad .$$

The Δ_{jmsp} and X_{jmsp} are totally symmetric expressions, [Cohen, 1976]

$$\Delta_{jmsp} = \delta_{jm}\delta_{sp} + \delta_{js}\delta_{mp} + \delta_{jp}\delta_{sm} \tag{52.1}$$

$$X_{jmsp} = x_j x_m \delta_{sp} + x_s x_p \delta_{jm} + x_j x_s \delta_{mp} + x_m x_p \delta_{js} + x_j x_p \delta_{ms} + x_m x_s \delta_{jp} \tag{52.2}$$

with the properties

$$\Delta_{jm} = \Delta_{jmss} = 5\delta_{jm} \qquad , \qquad \Delta_{ss} = 15 \tag{52.3}$$

$$X_{jm} = X_{jmss} = R^2\delta_{jm} + 7x_j x_m \qquad , \qquad X_{ss} = 10R^2 \tag{52.4}$$

Therefore, the first curly braces in Eq. (51.2) vanishes if contracted.

Since it may be verified that, for $R \to 0$,

$$\phi_n^{(p)}(R) \sim \frac{\kappa^2 - k^2}{R^{2n-1}} \qquad p = 0$$

$$\sim \frac{\kappa^p - k^p}{R^{2n+1}} \qquad p \neq 0$$

it is seen that $G_{jm,s}(R) \sim R^{-2}$ and $G_{jm,sp} \sim R^{-3}$.

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Equation (28) may be differentiated to obtain

$$u_{m,p}(\underline{r}) = u_{m,p}^{(o)}(\underline{r}) + \omega^2 \int G_{mj,p}(\underline{r}-\underline{r'})\Delta\rho(\underline{r'})u_j(\underline{r'})d\underline{r'}$$

$$+ \int G_{mj,sp}(\underline{r}-\underline{r'})\left\{\Delta\lambda(\underline{r'})\delta_{js}\varepsilon(\underline{r'}) + 2\Delta\mu(r')\varepsilon_{js}(\underline{r'})\right\}d\underline{r'} \quad . \tag{52}$$

We define symmetrized functions

$$\bar{G}_{j;mp}(R) = \bar{G}_{mp;j}(R) = \frac{1}{2}\left[G_{mj,p}(R) + G_{pj,m}(R)\right]$$

$$= \frac{1}{4\pi i\rho\omega^2}\left[-\frac{\kappa^4}{2R}h_1(\kappa R)(x_p\delta_{mj} + x_m\delta_{pj})\right.$$

$$\left. + (x_j\delta_{mp} + x_m\delta_{jp} + x_p\delta_{jm})\phi_2^{(o)}(R) - x_j x_m x_p \phi_3^{(o)}(R)\right]$$

$$\tag{53}$$

and, with the delta-function component omitted,

$$\bar{G}_{mp;js}(R) = \frac{1}{2}\left[G_{mp;j,s}(R) + G_{mp;s,j}(R)\right]$$

$$= \frac{1}{4\pi i\rho\omega^2}\left[-\frac{\kappa^4}{2R}h_1(\kappa R)(\delta_{ps}\delta_{jm} + \delta_{pj}\delta_{ms})\right.$$

$$+ \frac{\kappa^5}{4R^2}h_2(\kappa R)(\delta_{jm}x_p x_s + \delta_{jp}x_m x_s + \delta_{ms}x_j x_p + \delta_{ps}x_m x_j)$$

$$\left. + \Delta_{jmsp}\phi_2^{(o)}(R) - X_{jmsp}\phi_3^{(o)}(R) + x_j x_m x_s x_p \phi_4^{(o)}(R)\right] \tag{54}$$

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We now introduce an operator notation which can be defined in an obvious
way by

$$\mathscr{G}_{mj}u_j \equiv \omega^2 \int G_{mj}(\underline{r}-\underline{r}')\Delta\rho(\underline{r}')u_j(\underline{r}')d\underline{r}' \tag{55.1}$$

$$\mathscr{G}_{m;js}\varepsilon_{js} \equiv \int \bar{G}_{m;pn}(\underline{r}-\underline{r}')\Delta c_{pnjs}(\underline{r}')\varepsilon_{js}(\underline{r}')d\underline{r}' \tag{55.2}$$

$$\mathscr{G}_{mp;j}u_j \equiv \omega^2 \int \bar{G}_{mp;j}(\underline{r}-\underline{r}')\Delta\rho(\underline{r}')u_j(\underline{r}')d\underline{r}' \tag{55.3}$$

$$\mathscr{G}_{mp;js}\varepsilon_{js} \equiv \int \bar{G}_{mp;nq}(\underline{r}-\underline{r}')\Delta c_{nqjs}(\underline{r}')\varepsilon_{js}(\underline{r}')d\underline{r}' \tag{55.4}$$

Then Eqs. (28) and (52) lead to

$$u_m(\underline{r}) = u_m^o(\underline{r}) + \mathscr{G}_{mj}u_j + \mathscr{G}_{m;js}\varepsilon_{js} \tag{56.1}$$

$$\varepsilon_{mp}(\underline{r}) = \varepsilon_{mp}^o(\underline{r}) + \mathscr{G}_{mp;j}u_j + \mathscr{G}_{mp;js}\varepsilon_{js}$$

$$+ \frac{1}{15\rho\omega^2}\left[(\kappa^2 - k^2)\delta_{mp}\Delta c_{nnjs} - (3\kappa^2 + 2k^2)\Delta c_{mpjs}\right]\varepsilon_{js}(\underline{r}) \tag{56.2}$$

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where the final term in Eq. (56.2) arises from the delta function in Eq. (51). Introducing the shorthand notation $\mathcal{G}_{mp} = \mathcal{G}_{mp;j} u_j + \mathcal{G}_{mp;js} \varepsilon_{js}$ for the integral terms in Eq. (56.2) gives, formally, an algebraic matrix equation to be solved for $\varepsilon_{mp}(\underline{r})$. However, in the case where Δc_{mpjs} is isotropic this system can be diagonalized. In the isotropic case, which is the only one we shall explore in detail from this point on, Eq. (56.2) is

$$\varepsilon_{mp}(\underline{r}) = \varepsilon_{mp}^o(\underline{r}) + \mathcal{G}_{mp} - \frac{1}{15\rho\omega^2} \left[5\kappa^2 \Delta\lambda(\underline{r}) \varepsilon(\underline{r}) \delta_{mp} - 2(\kappa^2 - k^2) \Delta\mu(\underline{r}) \varepsilon(\underline{r}) \delta_{mp} \right.$$

$$\left. + 2(3\kappa^2 + 2k^2) \Delta\mu(\underline{r}) \varepsilon_{mp}(\underline{r}) \right] \tag{57}$$

Evaluating the dilatation and deviatrix of this expression gives

$$\varepsilon(r) = \varepsilon^o(\underline{r}) + \mathcal{G}_{ss} - a(r)\varepsilon(r) \quad ; \quad a(r) = \frac{\Delta\lambda(r) + \frac{2}{3}\Delta\mu(r)}{\lambda + 2\mu} \tag{58.1}$$

$$E_{mp}(\underline{r}) = E_{mp}^o(\underline{r}) + \mathcal{G}_{mp} - \frac{1}{3}\mathcal{G}_{ss}\delta_{mp} - b(r)E_{mp}(\underline{r})$$

$$b(r) = \frac{2(3\lambda+8\mu)}{15\mu(\lambda+2\mu)} \Delta\mu(r) \tag{58.2}$$

and hence

$$\varepsilon_{mp}(\underline{r}) = \frac{1}{1+b(r)} \left[\varepsilon_{mp}^o(r) + \mathcal{G}_{mp} + \frac{1}{3}\alpha(r)\left(\varepsilon^o(r) + \mathcal{G}_{ss}\right)\delta_{mp} \right]$$

$$\alpha(r) = \frac{b(r) - a(r)}{1 + a(r)} \tag{59}$$

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The integral equation (59) is not yet in a convenient form for a Neumann expansion since $\mathscr{G}_{mp}$ still involves integrals which do not vanish in the limit as the characteristic dimension of the scatter goes to zero, although for the special case of a sphere (because of its symmetry) those integrals vanish. In that one case, then, using $\varepsilon_{mp} = (1+b)^{-1}\left(\varepsilon_{mp}^{o}(\underline{r}) + \frac{1}{3}\,\alpha\varepsilon^{o}(\underline{r})\delta_{mp}\right)$ in Eq. (26) gives the "corrected" Born approximation [Cohen, 1976; Mal and Knopoff, 1976].

In order to evaluate the complete first order approximation to the strain field inside the scatterer it is necessary to evaluate the lowest order contributions of the integrals $\mathscr{G}_{mp}$, and hence, equivalently, the long-wavelength limit (static) Green's function. If we assume the long-wavelength limit, then $u_{m}(\underline{r})$ will be a slowly varying function of position within the scatterer and we may use a mean-value theorem to remove $u_{m}(\underline{r})$ from the integrals. Furthermore, we may use Eshelby's result that ε_{mp} is constant inside a homogeneous ellipsoid in the static limit and remove it from the integral in that case without resort to any approximation. Then, keeping only those terms which can contribute in lowest order, we may write Eqs. (58.1) and (58.2) in a convenient symmetric form:

$$(1+a)\varepsilon(\underline{r}) = \varepsilon^{o}(\underline{r}) + \left(\Delta\lambda + \frac{2}{3}\,\Delta\mu\right)g\varepsilon(\underline{r}) + 2\Delta\mu g_{js}E_{js}(\underline{r}) \tag{60.1}$$

$$(1+b)E_{mp}(\underline{r}) = E_{mp}^{o}(\underline{r}) + \left(\Delta\lambda + \frac{2}{3}\,\Delta\mu\right)g_{mp}\varepsilon(\underline{r}) + 2\Delta\mu g_{mp;js}E_{js}(\underline{r}) \tag{60.2}$$

where the integrals are defined by

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$$g = \int \bar{G}_{pp;jj}(\underline{r}-\underline{r}')d\underline{r}' \tag{61.1}$$

$$g_{mp} = \int \bar{G}_{mp;jj}(\underline{r}-\underline{r}')d\underline{r}' - \frac{1}{3} g\delta_{mp} \tag{61.2}$$

$$g_{mp;js} = \int \bar{G}_{mp;js}(\underline{r}-\underline{r}')d\underline{r}' \quad . \tag{61.3}$$

We note that it is possible to write g_{js} in Eq. (60.1) as the same functional expression as g_{mp} in Eq. (60.2) because of the fact that E_{js} has zero trace. Therefore, adding the term $-\frac{1}{3}\delta_{js}\bar{G}_{pp;nn}(\underline{r}-\underline{r}')E_{js}(\underline{r}')$ in Eq. (58.1) adds nothing, but allows the symmetric structure of the equations. In fact, since $\bar{G}_{pp;nn}(R) = -(k^2/4\pi\rho\omega^2 R)e^{-ikR}$ is a lower order singularity, its contribution to g_{js} is $O(k^2a^2)$ and hence vanishes to second order. One must be cautious here; $G_{pp;nn}(R)$ still contains a delta function, $-\delta(R)/(\lambda+2\mu)$, which is, of course, not a lower order singularity and cannot be neglected.

The integrals must now be evaluated in the long-wavelength limit. Expanding the Bessel function gives us

$$G_{jm} = \frac{1}{4\pi\rho\omega^2 R}\left[\frac{1}{3}(2\kappa^2 + k^2)\delta_{jm} + \left(\frac{3x_j x_m}{R^2} - \delta_{jm}\right)\frac{\kappa^2 - k^2}{6} + \ldots\right]$$

$$= \frac{1}{8\pi\rho\omega^2 R}\left[\kappa^2\left(\delta_{jm} + \frac{x_j x_m}{R^2}\right) + k^2\left(\delta_{jm} - \frac{x_j x_m}{R^2}\right) + \ldots\right] \tag{62.1}$$

or, in the long-wavelength limit,

$$G_{jm}^o = \frac{1}{8\pi\mu(\lambda+2\mu)R}\left[(\lambda+3\mu)\delta_{jm} + (\lambda+\mu)\frac{x_j x_m}{R^2}\right] \quad . \tag{62.2}$$

$$\bar{G}_{js;m}(R) = \frac{1}{8\pi\rho\omega^2 R^3}\left[\kappa^2 x_m\left(\delta_{js} - 3\frac{x_j x_s}{R^2}\right) + k^2\left(3\frac{x_j x_s x_m}{R^2} - x_j\delta_{ms} - x_s\delta_{mj} - x_m\delta_{js}\right) + \cdots\right] \tag{63.1}$$

$$\bar{G}_{js;m}^o = \frac{-1}{8\pi\mu(\lambda+2\mu)R^3}\left[\mu(\delta_{jm}x_s + \delta_{sm}x_j) + (\lambda+\mu)x_m\left(3\frac{x_j x_s}{R^2} - \delta_{js}\right)\right] \tag{63.2}$$

and

$$\bar{G}_{mp;js}(R) = \frac{1}{16\pi\rho\omega^2 R^3}\left[\kappa^2\left\{2\delta_{mp}\delta_{js} - \frac{3}{R^2}(x_m x_p\delta_{js} + x_j x_s\delta_{mp} + X_{mpjs}) + \frac{30 x_m x_p x_j x_s}{R^4}\right.\right.$$

$$\left.\left. -2k^2\left\{\Delta_{mpjs} - 3\frac{X_{mpjs}}{R^2} + \frac{15 x_m x_p x_j x_s}{R^4}\right\} + \cdots\right] \tag{64.1}$$

$$\bar{G}_{mp;js}^o = \frac{-1}{16\pi\mu(\lambda+2\mu)R^3}\left[2\mu\Delta_{mpjs} + \frac{3\lambda}{R^2}X_{mpjs} - \frac{30(\lambda+\mu)}{R^4}x_m x_p x_j x_s\right.$$

$$\left. + (\lambda+2\mu)\left\{\delta_{mp}\left(3\frac{x_j x_s}{R^2} - \delta_{js}\right) + \left(3\frac{x_m x_p}{R^2} - \delta_{mp}\right)\delta_{js}\right\}\right] \tag{64.2}$$

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In these expressions only the lowest order terms have been retained; the Green's function is, to this order, the Green's function for the elastostatic problem [Eshelby, 1961; Love, 1934].

We now use the divergence theorem to convert these integrals into surface integrals and evaluate them at the center of the ellipsoid. This transformation, however, recaptures the effect of the delta function in $G_{mp;js}$ and it will therefore be necessary to consider a small sphere which excludes the origin as part of the surface of integration or to appropriately modify the equation in order to avoid including this contribution twice. We shall choose the latter approach and merely insert the modifications later. Then

$$g_{mp;js} = \frac{1}{2} \int \left[\bar{G}^o_{mp;j}(\underline{r}')n_s + \bar{G}^o_{mp;s}(\underline{r}')n_j \right] dS'$$

$$= \frac{-1}{16\pi\mu(\lambda+2\mu)} \int \left[\mu(\delta_{jm}x_p n_s + \delta_{pj}x_m n_s + \delta_{ps}x_m n_j + \delta_{ms}x_p n_j) \right.$$

$$\left. + (\lambda+\mu)(n_j x_s + n_s x_j)\left(\frac{3x_m x_p}{R^2} - \delta_{mp}\right) \right] \frac{dS'}{R^3} \quad . \quad (65)$$

For an ellipsoid defined by

$$x_1 = a_1 \sin\theta\cos\phi$$

$$x_2 = a_2 \sin\theta\sin\phi \quad , \quad R^2 = a_3^2 + \left(a_1^2 - a_3^2 + (a_2^2 - a_1^2)\sin^2\phi\right)\sin^2\theta \quad (66)$$

$$x_3 = a_3 \cos\theta$$

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the normal surface vectors are given by

$$n_j dS = \frac{a_1 a_2 a_3}{a_j^2} x_j \sin\theta d\theta d\phi \quad , \tag{67}$$

and $g_{mp;js}$ can be expressed in terms of two classes of integrals

$$J_{mp} = \frac{\left(a_1 a_2 a_3\right)^{1/3}}{4\pi} \int \frac{x_m x_p \sin\theta d\theta d\phi}{R^3} \quad , \tag{68.1}$$

$$K_{mpjs} = \frac{\left(a_1 a_2 a_3\right)^{1/3}}{4\pi} \int \frac{x_m x_p x_j x_s \sin\theta d\theta d\phi}{R^5} \quad . \tag{68.2}$$

From symmetry we find immediately that J_{mp} is diagonal and that there are only six independent non-zero components for K_{mpjs}. Using six-vector notation with $mp \rightarrow \alpha$, $js \rightarrow \beta$ we see that K_{mpjs} may be written

$$K_{mpjs} = K_{\alpha\beta} = \begin{pmatrix} K_1 & K_6 & K_5 & & & \\ K_6 & K_2 & K_4 & & 0 & \\ K_5 & K_4 & K_3 & & & \\ & & & K_4 & 0 & 0 \\ & 0 & & 0 & K_5 & 0 \\ & & & 0 & 0 & K_6 \end{pmatrix} \tag{69}$$

and therefore

$$g_{mp;js} = \frac{\left(a_1a_2a_3\right)^{2/3}}{4\mu(\lambda+2\mu)} \left[(\lambda+\mu)(\delta_{mp}J_{js} - 3K_{mpjs})\left(\frac{1}{a_j^2} + \frac{1}{a_s^2}\right)\right.$$

$$\left. - \mu\left(\delta_{jm}\frac{J_{ps}}{a_p a_s} + \delta_{pj}\frac{J_{ms}}{a_m a_s} + \delta_{ps}\frac{J_{mj}}{a_m a_j} + \delta_{ms}\frac{J_{pj}}{a_p a_j}\right)\right]$$

$$(70.1)$$

where

$$g_{\alpha\beta} = \frac{\left(a_1a_2a_3\right)^{2/3}}{2\mu(\lambda+2\mu)} \left[(\lambda+\mu)A_{\alpha\beta} + \mu B_{\alpha\beta}\right]$$

$$A_{\alpha\beta} = \begin{pmatrix} \dfrac{3K_1-J_{11}}{a_2^1} & \dfrac{3K_5-J_{22}}{a_2^2} & \dfrac{3K_5-J_{33}}{a_3^2} & & & \\[3mm] \dfrac{3K_6-J_{11}}{a_1^2} & \dfrac{3K_2-J_{22}}{a_2^2} & \dfrac{3K_4-J_{33}}{a_3^2} & & 0 & \\[3mm] \dfrac{3K_5-J_{11}}{a_1^2} & \dfrac{3K_4-J_{22}}{a_2^2} & \dfrac{3K_3-J_{33}}{a_3^2} & & & \\[3mm] & & & \dfrac{3K_4(a_2^2+a_3^2)}{2a_2^2a_3^2} & 0 & 0 \\[3mm] & 0 & & 0 & \dfrac{3K_5(a_1^2+a_3^2)}{2a_1^2a_2^2} & 0 \\[3mm] & & & 0 & 0 & \dfrac{3K_6(a_1^2+a_2^2)}{2a_1^2a_2^2} \end{pmatrix}$$

$$(70.2)$$

and

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$$
B_{\alpha\beta} = \begin{pmatrix}
J_{11}/a_1^2 & 0 & 0 & & & \\
& J_{22}/a_2^2 & 0 & & 0 & \\
0 & 0 & J_{33}/a_3^2 & & & \\
& & & \dfrac{J_{11}a_3^2 + J_{33}a_2^2}{4a_2^2 a_3^2} & 0 & 0 \\
& 0 & & 0 & \dfrac{J_{11}a_3^2 + J_{33}a_1^2}{4a_1^2 a_3^2} & 0 \\
& & & 0 & 0 & \dfrac{J_{11}a_2^2 + J_{22}a_1^2}{4a_1^2 a_2^2}
\end{pmatrix}
$$

$$(70.3)$$

Among the nine integrals there are three immediate relationships

$$
J_{11} = K_1 + K_6 + K_5
$$

$$
J_{22} = K_6 + K_2 + K_4 \tag{71.1}
$$

$$
J_{33} = K_5 + K_4 + K_3
$$

Since it is also clear from Eqs. (61.3) and (64.2) that $A_{\alpha\beta}$ is a symmetric matrix, the asymmetric form of Eq. (70.2) must be illusory and it can indeed be verified that

$$
3(a_2^2 - a_3^2)K_4 = a_2^2 J_{33} - a_3^2 J_{22}
$$

$$
3(a_3^2 - a_1^2)K_5 = a_3^2 J_{11} - a_1^2 J_{33} \tag{71.2}
$$

$$
3(a_1^2 - a_2^2)K_6 = a_1^2 J_{22} - a_2^2 J_{11}
$$

so that the six K_j may all be expressed in terms of the three J's.

One also finds

$$g_{mp} \equiv g_{mp;jj} = - \frac{1}{8\pi(\lambda+2\mu)} \int (x_m n_p + x_p n_m) \frac{dS'}{R^3}$$

$$= - \frac{1}{\lambda+2\mu} \left[\frac{\left(a_1 a_2 a_3\right)^{2/3}}{a_m a_p} J_{mp} - \frac{1}{3} \right] \delta_{mp} \tag{72}$$

where the second term in the bracket comes from subtracting out the delta function in $G_{mp;jj}$. We may then write Eq. (60) as*

$$E_\alpha = E_\alpha^o / (1 - 4\Delta\mu g_{\alpha\alpha}) \qquad\qquad \alpha = 4,5,6 \tag{73.1}$$

$$E_1 = E_1^o + 2\Delta\mu \left\{ (g_{11} - g_{13})E_1 + (g_{12} - g_{13})E_2 - \frac{1}{\lambda+2\mu} \left[\left(\frac{a_2 a_3}{a_1^2}\right)^{2/3} J_{11} - \frac{1}{3} \right]\varepsilon \right\}$$
$$\tag{73.2}$$

$$E_2 = E_2^0 + 2\Delta\mu \left\{ (g_{12} - g_{23})E_1 + (g_{22} - g_{23})E_2 - \frac{1}{\lambda+2\mu} \left[\left(\frac{a_1 a_3}{a_2^2}\right)^{2/3} J_{22} - \frac{1}{3} \right]\varepsilon \right\}$$
$$\tag{73.3}$$

---

*It should be remembered that in 6-vector space a scalar product is defined by $A \cdot B = A_1 B_1 + A_2 B_2 + A_3 B_3 + 2(A_4 B_4 + A_5 B_5 + A_6 B_6)$ in order to account for the symmetrization of the subscripts ij and ji into a single component.

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$$(1+a)\varepsilon = \varepsilon^o - \Delta Kc \left[\left(\frac{J_{11}}{a_1^2} - \frac{J_{33}}{a_3^2} \right) E_1 + \left(\frac{J_{22}}{a_2^2} - \frac{J_{33}}{a_3^2} \right) E_2 \right] \tag{73.4}$$

where $c \equiv (a_1 a_2 a_3)^{2/3}/(\lambda+2\mu)$ and $\Delta K = \Delta\lambda + \frac{2}{3}\Delta\mu$, and use has been made explicitly of the trace condition

$$E_1 + E_2 + E_3 = 0 \quad . \tag{74}$$

For a spherical scatterer, $a_1 = a_2 = a_3 = a$ and $J_{11} = J_{22} = J_{33} = \frac{1}{3}$, $K_1 = K_2 = K_3 = \frac{1}{5}$, $K_4 = K_5 = K_6 = \frac{1}{15}$. Then we find, from Eq. (70) that $-2\Delta\mu(g_{11} - g_{13}) = -2\Delta\mu(g_{22} - g_{23})$ is just the "shielding" factor, b, previously obtained from the delta function in Eq. (58.2) and that this corresponds also to $-4\Delta\mu g_{44} = -4\Delta\mu g_{55} = -4\Delta\mu g_{66} = b$. Thus, for a spherical scatterer, the only contributions to the renormalization of the internal strain field are the shielding factors a and b which arise from the delta function singularity in Green's function. For an arbitrary ellipsoid, or for any more general shape, we have now explicitly exhibited the volume "polarization" contributions to this renormalization.

Ellipsoid of Revolution

For an ellipsoidal scatterer with $a_1 = a_2$ we then have $J_{11} = J_{22}$, $K_1 = K_2 = 3K_6$, $K_4 = K_5$, $g_{11} = g_{22}$, and $g_{13} = g_{23}$. We also find

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$$J_{11} = J_{22} = \frac{(a_1/a_3)^{2/3}}{2(a_3^2 - a_1^2)} \left[a_3^2 - a_1^2 I(a_1/a_3) \right] \quad , \quad \begin{cases} \dfrac{\sec^{-1}u}{\sqrt{u^2 - 1}} & u > 1 \quad (75.1) \\ \\ \dfrac{\text{sech}^{-1}u}{\sqrt{1 - u^2}} & u < 1 \quad (75.2) \end{cases}$$

$$I(u) =$$

$$J_{33} = \frac{(a_1/a_3)^{2/3}a_3^2}{(a_3^2 - a_1^2)} \left[I(a_1/a_3) - 1 \right]$$

$$K_4 = K_5 = \frac{J_{11}a_3^2 - J_{33}a_1^2}{3(a_3^2 - a_1^2)} = \frac{(a_1/a_3)^{2/3}a_3^2}{6(a_3^2 - a_1^2)^2} \left[a_3^2 + 2a_1^2 - 3a_1^2 I(a_1/a_3) \right] \qquad (75.3)$$

$$K_1 = K_2 = 3K_6 = \frac{3}{4}(J_{11} - K_4) = \frac{(a_1/a_3)^{2/3}}{8(a_3^2 - a_1^2)^2} \left[2a_3^4 - 5a_1^2a_3^2 + 3a_1^4 I(a_1/a_3) \right]$$

$$(75.4)$$

$$K_3 = J_{33} - 2K_4 = \frac{(a_1/a_3)^{2/3}}{3(a_3^2 - a_1^2)^2} \left[3a_3^2 I(a_1/a_3) - 4a_3^2 + a_1^2 \right] \qquad (75.5)$$

Then, from Eq. (70.2)

$$g_{11} = g_{22} = \frac{(a_1/a_3)^{2/3}}{2\mu(\lambda+2\mu)} \left[(\lambda-u)J_{11} - 3(\lambda+\mu)K_1 \right] \qquad (76.1)$$

$$g_{12} = \frac{(a_3/a_1)^{2/3}(\lambda+\mu)}{2\mu(\lambda+2\mu)} (J_{11} - \dot{K}_1) \qquad (76.2)$$

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$$g_{13} = g_{23} = \frac{\left(a_3/a_1\right)^{2/3}(\lambda+\mu)}{2\mu(\lambda+2\mu)} \ (J_{11} - 3K_4) \tag{76.3}$$

$$g_{33} = \frac{\left(a_1/a_3\right)^{2/3}}{2\mu(\lambda+2\mu)} \ \left[(\lambda-\mu)J_{33} - 3(\lambda+\mu)K_3\right] \tag{76.4}$$

$$g_{44} = g_{55} = - \frac{1}{2\mu(\lambda+2\mu)} \left[\left(\frac{a_3}{a_1}\right)^{2/3}\left\{\mu J_{11} + 3(\lambda+\mu)K_4\right\} + \left(\frac{a_1}{a_3}\right)^{4/3}\left\{\mu J_{33} + 3(\lambda+\mu)K_4\right\}\right] \tag{76.5}$$

$$g_{66} = \frac{\left(a_3/a_1\right)^{2/3}}{2\mu(\lambda+2\mu)} \ \left[\mu J_{11} + 3(\lambda+\mu)K_6\right] \tag{76.6}$$

Using these expressions reduces Eq. (73) to a simpler form, which may be solved rather directly to give

$$\varepsilon = \frac{\varepsilon^o(1 + \gamma^+) - \gamma c\Delta K(E_1^o + E_2^o)}{(1+a)(1 + \gamma^+) - 4\beta\gamma c\Delta\mu\Delta K} \tag{77.1}$$

where

$$\gamma = \frac{J_{11}}{a_1^2} - \frac{J_{33}}{a_3^2}$$

$$\beta = \frac{\left(a_3/a_1\right)^{2/3}J_{11} - 1/3}{\lambda+2\mu}$$

$$\gamma^+ = 2\Delta\mu(2g_{13} - g_{11} - g_{12})$$

$$E_1 + E_2 = \frac{(1+a)(E_1^o + E_2^o) - 4\beta\Delta\mu\epsilon^o}{(1+a)(1 + \gamma^+) - 4\beta\gamma c\Delta\mu\Delta K} \tag{77.2}$$

$$E_1 - E_2 = \frac{E_1^o - E_2^o}{1 - 2\Delta\mu(g_{11} - g_{12})} \tag{77.3}$$

These equations then provide the shielding factors for the extended Born approximation. Given the incident strain field $\epsilon_{pm}^o(\underline{r}) = E_{mp}^o(r) + \frac{1}{3}\delta_{mp}\epsilon^o(\underline{r})$ we use Eqs. (73.1) and (77.1)-(77.3) to provide an accurate first approximation to insert into Eqs. (25) and (26) for the scattered-wave amplitude or Eqs. (26) and (38) for the differential scattering cross section, noting however that the components of the displacement and the strain considered here are specifically written with respect to a coordinate system corresponding to the principal axes of the ellipsoid.

The important feature of this solution is that, to this level of approximation, the general structure of the scattering vector U_j given in Eqs. (42)-(45) above is unchanged. The density component, that involving $\Delta p/p$ is exactly as given in those equations. The dilatation and shear components, involving f_d and f_μ will still involve the Fourier transform of the scattering volume and the shielding factors given in Eqs. (77.1)-(77.3). In addition, the purely geometrical aspects of the scattering law, (those due to the projection operators $\Omega_\mu\Omega_s$ and $\delta_{\mu s} - \Omega_\mu\Omega_s$ of the incident and scattered wave directions) will contribute no more than an "isotropic" component and components proportional to the second order spherical harmonics, and therefore no more than terms of the form $a_0 + a_2\cos 2\theta$ or $\sin 2\theta$ in the scattering angle, θ. The amplitudes of the various components

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will, however, also depend on the orientation of the scattering ellipsoid
with respect to the plane of scattering.

The full implementation of this stage of the analysis has not been
completed.

F. Variational Theorems

Although integral equations are often much more difficult to solve than
the corresponding differential equations, they are often the more convenient
starting point for approximate solutions. This is particularly true when a
complete solution is not needed. In the scattering of elastic waves one has
an ideal example of this sort of approximation. We are not interested in a
complete solution of Eq. (3) or even a complete solution, for all $\underline{r}$, of Eq.
(28). What we are interested in is the vector $\underline{U}(q)$ which is given in Eq. (26)
as an integral over the volume of the scatterer. This situation lends itself
ideally to attack by variational methods. To establish a methodology for such
an approach[*] we consider an unknown function $u(x)$ as the solution of the
integral equation

$$u(x) = u^{o}(\underline{x}) + \int H(x;x')u(x')dx' \qquad (78)$$

and for which we want to evaluate the integral

$$J = \int f(x)u)x)dx \qquad (79)$$

In order to define a variational approximation for J we introduce an
adjoint equation

[*]A different approach to the formulation of a variation equation has been given
recently by G. S. Kino (Kino, 1976].

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$$v(x) = f(x) + \int H^+(x;x')v(x')dx' \qquad (80)$$

where $H^+(x;x')$ is the adjoint operator, such that for any two arbitrary sufficiently continuous functions, $f(x)$ and $g(x)$,

$$\iint g(x)H(x;x')f(x')dxdx' = \iint f(x')H^+(x';x)g(x)dxdx'_i \quad ,$$

When $H(x;x)$ involves only algebraic factors but no operators (such as $\partial/\partial x$, $\partial/\partial x'$, etc.), $H^+(x';x) = H(x;x')$.

Now if we write

$$Q[\tilde{u},\tilde{v}] = \int u^o(x)\tilde{v}(x)dx + \int \tilde{\tilde{u}}(x)f(x)dx$$

$$- \int \tilde{u}(x)\tilde{v}(x)dx + \iint \tilde{v}(x)H(x;x')\tilde{u}(x')dxdx' \qquad (81)$$

we have a variational formulation for the integral J. If the function $\tilde{u}$ is equal to u, the solution of Eq. (78), we find

$$Q[u,\tilde{v}] = \int u(x)f(x)dx = J \qquad (82.1)$$

and if $\tilde{v}$ is equal to v, the solution of Eq. (80), we find

$$Q[\tilde{u}, v] = \int u^o(x) v(x) dx$$

$$= \int u(x) v(x) dx - \iint H(x; x') u(x') v(x) dx' dx$$

$$= \int f(x) u(x) dx = J \tag{82.2}$$

Thus, if $\tilde{u}$ and $\tilde{v}$ are approximations to the exact solutions of Eqs. (78) and (80), $Q[\tilde{u}, \tilde{v}]$ is a second order accurate approximation to J. If we set $\tilde{u} = u + \eta$, $\tilde{v} = v + \xi$ we find

$$Q[u+\eta, v+\xi] = J + \iint \xi(x) H(x; x') \eta(x') dx dx'$$

$$- \int \xi(x) \eta(x) dx \tag{83}$$

so that the error is dependent on the product of the errors of the approximating functions.

The variational expression Eq. (81) may be easily improved for a scalar function. Let us write $\tilde{u} = c_1 u_1$, $\tilde{v} = c_2 v_1$ where c_1 and c_2 are constants. Thus if we can assume a shape for u and v we may use the variational process to define the optimum amplitudes for those functions. Then, with an obvious shorthand,

$$Q[c_1 u_1, c_2 v_1] = c_2(u^o, v_1) + c_1(u_1, f) - c_1 c_2(u_1, v_1)$$

$$+ c_1 c_2(v_1 H u_1) \tag{84}$$

We now differentiate Q with respect to c_1 and c_2 and set these expressions to zero.

$$(u_1, f) = c_2(u_1, v_1) - c_2(u_1 H v_1)$$

$$(u^o, v_1) = c_1(u_1, v_1) - c_1(u_1 H v_1)$$

and from this we find, [Goldstein and Cohen, 1962]

$$J(u_1, v_1) = \frac{(u_1, f)(u^o, v_1)}{(u_1, v_1) - (v_1 H u_1)} \tag{85}$$

Equation (85) then represents a formulation of the variational problem which is homogeneous of degree zero in the trial functions and hence independent of their normalization.

To apply this formalism to elastic scattering we write Eqs. (28) and (26) in terms of a nine-component vector, $\underline{z} = (u_m, \varepsilon_\alpha, E_1, E_2, \varepsilon)$, $m = 1,2,3$, $\alpha = 4,5,6$, with, of course, $E_3 = -E_1 - E_2$, and $\varepsilon_\beta = \frac{1}{3}\varepsilon + E_\beta$, $\beta = 1,2,3$. Equations (26) and (28) are then converted into expressions of the form

$$U_j(\underline{q}) = \int f_s(\underline{r}; \underline{q}, j) z_s(\underline{r}) dr \tag{86.1}$$

and

$$z_s(\underline{r}) = z_s^o(\underline{r}) + \int H_{sp}(\underline{r} - \underline{r}') z_p(\underline{r}') d\underline{r}' \tag{86.2}$$

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where

$$z_s^o(r) = \left(u_m^o(r) \quad , \quad \frac{\varepsilon_\alpha^o(r)}{1+b} \quad , \quad \frac{E_1^o(r)}{1+b} \quad , \quad \frac{E_2^o(r)}{1+b} \quad , \quad \frac{\varepsilon^o(r)}{1+a} \right) \quad . \tag{86.3}$$

The expressions for the matrix elements H_{sp} can be written out explicitly by comparing the form of (86.2) with Eqs. (55)-(59). Although this identification is straightforward, the explicit expression for 9×9 matrix is cumbersome and need not be given here. The vector $f_s(\underline{r};\underline{q},j)$ may also be found from Eq. (26). For f_s we find

$$f_s(\underline{r};\underline{q};1) = \frac{e^{i\underline{q}\cdot\underline{r}}}{4\pi} \left(\frac{\Delta\rho}{\rho} \;,0,0,0, \; - \frac{2i\Delta\mu}{\rho\omega^3}\, q_3, \; - \frac{2i\Delta\mu}{\rho\omega^2}\, q_2, \; - \frac{2i\Delta\mu}{\rho\omega^2}\, q_1,0, \; - \frac{i\Delta K}{\rho\omega^2}\, q_1 \right)$$

$$\tag{87.1}$$

$$f_s(\underline{r};\underline{q},2) = \frac{e^{i\underline{q}\cdot\underline{r}}}{4\pi} \left(0, \; \frac{\Delta\rho}{\rho} \;,0, \; - \frac{2i\Delta\mu}{\rho\omega^2}\, q_3,0, \; - \frac{2i\Delta\mu}{\rho\omega^2}\, q_1,0, \; - \frac{2i\Delta\mu}{\rho\omega^2}\, q_2, \; - \frac{i\Delta K}{\rho\omega^2}\, q_2 \right)$$

$$\tag{87.2}$$

$$f_s(\underline{r};\underline{q},3) = \frac{e^{i\underline{q}\cdot\underline{r}}}{4\pi} \left(0,0, \; \frac{\Delta\rho}{\rho} \;, \; - \frac{2i\Delta\mu}{\rho\omega^2}\, q_2, \; - \frac{2i\Delta\mu}{\rho\omega^2}\, q_1,0, \; \frac{2i\Delta\mu}{\rho\omega^2}\, q_3, \; \frac{2i\Delta\mu}{\rho\omega^2}\, q_3, \; - \frac{i\Delta K}{\rho\omega^2}\, q_3 \right)$$

$$\tag{87.3}$$

The generalization of Eq. (85) is straightforward. One introduces two trial functions (vectors), $\tilde{u}_s(\underline{r})$ and $\tilde{v}_s(\underline{r})$ from which one constructs two vectors W and W^+ with components

$$w_s = \int f_s(\underline{r};\underline{q},j)\tilde{u}_s(\underline{r})d\underline{r} \tag{88.1}$$

(not summed on s or p)

$$w_p^+ = \int z_p^o(\underline{r})\tilde{v}_p(\underline{r})d\underline{r} \tag{88.2}$$

and a 9×9 matrix W_o with components

$$w_{pm}^o = \int \tilde{v}_p(\underline{r})\tilde{u}_m(\underline{r})d\underline{r}\delta_{pm} - \iint \tilde{v}_p(\underline{r})H_{pm}(\underline{r}-\underline{r}')\tilde{u}_m(\underline{r}')d\underline{r}'d\underline{r} \quad . \tag{88.3}$$

Then the variational expression for U_j becomes

$$U_j[\tilde{u},\tilde{v}] = WW_o^{-1}W^+$$

The variational solution for the scattering function $U_j(\underline{q})$ is to be found by introducing an appropriate set of free parameters into the trial functions $\tilde{u}$ and $\tilde{v}$ and to determine the "best" values of these free parameters by requiring that U_j be stationary with respect to variation of those parameters. In fact, since Eq. (89) is already based on an optimization procedure with respect to a set of linear parameters, a significant improvement in the evaluation of U_j comes from inserting an appropriate first approximation for $\tilde{u}$ and $\tilde{v}$ into Eq. (89) and evaluating the expression with no additional free parameters. Thus we can use $\tilde{u}_s(\underline{r}) = z_s^o(\underline{r})$ and $\tilde{v}_s(\underline{r}) = f_s(\underline{r})$ as a first approximation. The integrals involved are essentially just those required for the Born approximation (i.e., Fourier transforms with respect to the change in propagation vector). If the wavelengths are not too short with respect to the dimensions of the

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scatterer the Green's function in $H_{pm}(\underline{r}-\underline{r}')$ in Eq. (89) may be expanded in powers of the frequency squared. The required integrals may then be evaluated in a straightforward manner.

When one uses $\tilde{u}_s(\underline{r}) = z_s^o(\underline{r})$ and $\tilde{v}_s(\underline{r}) = f_s(\underline{r})$ one obtains the $w_s(\underline{r}) = w_s^+(\underline{r})$ which allows the "corrected" Born approximation for the differential scattering cross section, Eq. (38.1), to be written in the form

$$S_\ell(\underline{\Omega}) = q^{in}k \left| \sum_j q_j^{sc} \sum_{s=1}^{9} w_s(\underline{q}^{sc},j) \right|^2 \tag{90.1}$$

and

$$S_t(\underline{\Omega}) = q^{in}\kappa \left\{ \kappa^2 \left| \sum_j \sum_s w_s(\underline{q}^{sc},j) \right|^2 - \left| \sum_j q_j^{sc} \sum_s w_s(\underline{q}^{sc},j) \right|^2 \right\} . \tag{90.2}$$

The lowest order variational calculation leads, for example, instead of Eq. (90.1), to the expression

$$S_\ell(\underline{\Omega}) = q^{in}k \left| \sum_j q_j^{sc} \sum_{s=1}^{9} w_s(\underline{q}^{sc};j)\phi_s \right|^2 \tag{91.1}$$

where

$$\phi_s = \sum_{p=1}^{q} \left(W_o^{-1} \right)_{sp} w_p . \tag{91.2}$$

In the lowest order, long-wavelength, approximation to $\underline{W}_o$, the correction, ϕ_s, is exactly equivalent to the shielding factors calculated above in Eq. (77).

The accuracy of this variational formulation when applied to ellipsoidal scatterers can be judged by comparing the variational calculation for a spherical scatterer with the available exact solution.

IV. PUBLICATIONS AND PRESENTATIONS

A. Presentations

1. B. R. Tittmann, "Mode Conversion and Angular Dependence for Scattering from Voids in Solids," IEEE Ultrasonics Symposium, Los Angeles, 1975, N-5.

2. B. R. Tittmann, "Elastic Wave Scattering from Spherical Cavities," Acoust. Soc. America, 90th Meeting, San Francisco, 1975, III-5.

3. B. R. Tittmann, "Elastic Wave Scattering from Ellipsoidal Cavities," presented at Fall Meeting of the Acoustical Society of America, November 15-19, 1976, San Diego, Calif. (Abstract appears in J. Acoustical Soc. Amer. 60, Supplement 1, S53 (1976).)

B. Publications

1. B. R. Tittmann, "Ultrasonic Scattering Studies for Failure Prediction," IEEE Trans. Sonics and Ultrasonics SU-24, 74 (1976).

2. E. Richard Cohen and B. R. Tittmann, "Analysis of Ultrasonic Wave Scattering for Characterization of Defects in Solids," Science Center, Rockwell International, Technical Report, SCTR-75-12 (December, 1975).

3. B. R. Tittmann, "Mode Conversion and Angular Dependence for Scattering from Voids in Solids," Ultrasonics Symposium Proc., IEE Cat. No. 75, CHO994-4SU, p. 111 (1975).

4. B. R. Tittmann, E. Richard Cohen, and J. M. Richardson, "Scattering of Longitudinal Waves Incident on a Cavity in a Solid," to be published in J. Acoustical Soc. Amer.

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5. E. Richard Cohen, "Analysis of Ultrasonic Scattering from Simply Shaped
    Objects," Proc. ARPA/AFML Review of Quantitative NDE, Science Center,
    Rockwell International (1975).

6. B. R. Tittmann and E. R. Cohen, "Scattering of Transverse Waves
    Incident on a Cavity in a Solid," (in preparation).

7. B. R. Tittmann and E. R. Cohen, "Scattering of Ultrasonic Waves from
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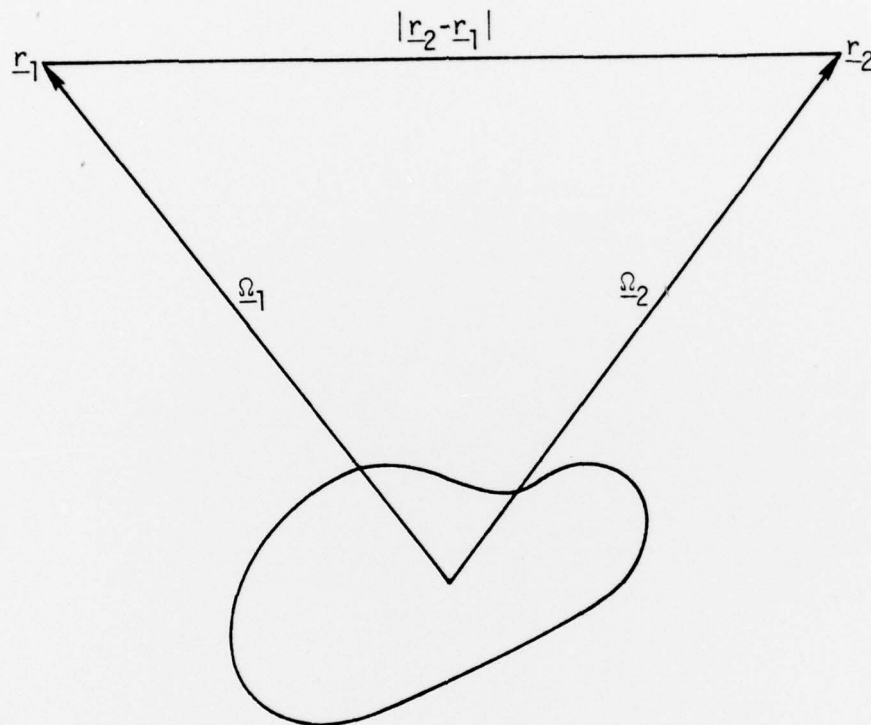Ying, C. F. and R. Truel (1956). J. Appl. Phys. $\underline{27}$, 1086.

Fig. 1 Geometry for establishing a reciprocity theorem for the scattering function U_s^o.